Braided Category \( \rightarrow \) Invariant of braids

Last time, we introduced braids as special cases of tangles.
Now, let's write down explicitly the generators and relations for isotopy classes of braids (both as groups and as a tensor category).

The Braid Group on \( n \) strands \( B_n \)

As a group, \( B_n \) is generated by \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \) (hence \( B_n \equiv \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \)) with the relations (when \( n \geq 3 \))

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i-j| > 1
\]

\( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \)

for \( 1 \leq i, j \leq n-1 \)

The Braid Category \( B \)

As a strict tensor category (same setup as for the Tangle Category), \( B \) is generated by \( \left\downarrow \right\downarrow \) and \( \left\downarrow \right\downarrow \)

together with the relations

\( \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \quad \text{and} \quad \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \quad \text{(R1)} \)

\( \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \quad \text{and} \quad \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \quad \text{(R2)} \)

\( \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \sim \left\downarrow \right\downarrow \quad \text{(R3)} \)

Comment:

1) We can deduce the generators and relations for \( B \) from the definition of \( B_n \) (In):

we save on the \# of generators : we can tensor them with \( \downarrow \) (identity) freely.
we need to add \( \downarrow \downarrow \) as a generator : unlike in a group where if \( \sigma \in G \),
\( \left(\text{R1}\right) \) as a relation then automatically \( \sigma^{-1} \in G \).

2) Compare generators & relations for \( B \) with those for \( T \):
exactly the ones without turns (\( U \) or \( N \)) survived.
Invariant of Braids

is a strict braided tensor functor from the Braid Category to another strict braided category.

Can be relaxed

Braided Tensor Functor \((F, \psi_0, \psi_2) : C \rightarrow D\)

\(C\) and \(D\) are braided tensor categories

\((F, \psi_0, \psi_2)\) is a tensor functor from \(C\) to \(D\)

\(\psi_2\) respects braiding \(C\)

Comment: If \(F\) is strict, then \(c_{F(v),F(v')} = F(c_{v,v'})\).

braiding \(C\) is preserved by \(F\).

Braided Category \(\mathcal{B}\) \rightarrow Invariant of Braids

Given a braided category \(C\) and an object \(V\) of \(C\), we want to construct a strict braided tensor functor from \(\mathcal{B}\) to \(C\).

\[
\text{Define } F(\cdot) := V.
\]

\[
\text{Ob}(\mathcal{B}) \xrightarrow{\text{Ob}(C)} \]

We can extend \(F\) uniquely to a strict braided tensor functor \(\mathcal{B} \rightarrow C\).

Exercise: verify.

In particular, \(F(\frac{\alpha}{\beta}) = c_{v,v}\) and \(F(\frac{\beta}{\alpha}) = c_{v,v}^{-1}\).

To compute \(F(B)\) for a given isotopy class of braids \(B\):

1. Write \(B\) as \(\circ\) and \(\otimes\) of the generating morphisms, say \(B = \circ_{i=1}^n (\otimes_{j=1}^m \mathcal{B}_{i,j})\).
2. Then by defn of a strict tensor functor, \(F(B) = \circ_{i=1}^n (\otimes_{j=1}^m F(\mathcal{B}_{i,j}))\).

E.g. \[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example1} \\
F \rightarrow \frac{F(X)}{F(X)} \frac{F(Y)}{F(Y)} \frac{F(Z)}{F(Z)} = (c_{v,v} \circ v) \circ (v \circ c_{v,v})
\end{array}
\]
Braided Hopf Algebra \( (H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R) \)

is a braided bialgebra with invertible antipode.

**Comment:** At this point, I am not sure whether it's better to define braided Hopf algebras with an invertible antipode or just an antipode. So I will always say "braided Hopf algebra with an invertible antipode" if I want to be invertible.

**Strict Tensor Category with Left Duality** \((C, \otimes, I, b, d)\)

\((C, \otimes, I)\) a strict tensor category

\(\left\{\begin{array}{l}
\text{For each } V \in \text{Ob}(C),
\exists \text{ a dual object } V^* \in \text{Ob}(C) \text{ and }
\begin{array}{l}
b_V : I \to V \otimes V^* \\
d_V : V^* \otimes V \to I
\end{array}
\in \text{Hom}(C)
\end{array}\right\}

\left( id(V) \otimes d(v)(b \otimes id(v)) = id(v) \quad \text{and} \quad (d(v) \otimes id(v^*)) (id \otimes b) = id(v^*) \right)

\[ u \quad \Rightarrow \quad \downarrow \quad = \quad \downarrow \quad \Rightarrow \quad v \]

\[ v \quad \Rightarrow \quad \downarrow \quad = \quad \downarrow \quad \Rightarrow \quad v \]

\[ \text{slightly ambiguous} \]

**Right Duality**

\[ \left\{\begin{array}{l}
\exists \text{ a dual object } V^* \in \text{Ob}(C) \text{ and }
\begin{array}{l}
b'_V : I \to V^* \otimes V \\
d'_V : V \otimes V^* \to I
\end{array}
\in \text{Hom}(C)
\end{array}\right\}

\left( d'(v) \otimes id(v) \right) \left( id(v) \otimes b' \right) = id(v) \quad \text{and} \quad (id \otimes d'(v)) \left( b' \otimes id(v^*) \right) = id(v^*) \right)

\[ v \quad \Rightarrow \quad \downarrow \quad = \quad \downarrow \quad \Rightarrow \quad v \]

Relation between Left & Right Duality

If \( C \) is autonomous, i.e. it has both left & right duality, then \( *(V^*) = V = *(V) \).

\[ 17.c \]
Strict Braided Category with Left/Right Duality

is a strict tensor category with left/right duality which is also braided.

Comment: As we saw in the definitions above, the universal R-matrix and the antipode can be introduced separately for a bialgebra, so can the braiding and duality in a tensor category.

We already saw how $R$ gives rise to a braiding in $A$-Mod.

universal R-matrix of a braided bialgebra $A$

We will now describe how $S$ gives a duality structure to $A$-Mod$_F$.

antipode of Hopf alg. $A$ $\rightarrow$ finite-diml $A$-modules

Construction: Antipode $\rightarrow$ Duality

Let $A$ be a Hopf algebra with antipode $S$.

For $V \in A$-Mod$_F$ with a basis $\{v_i, \ldots, v_n \}$, consider its dual vector space $\text{Hom}(V, k)$ with the dual basis $\{v^i, \ldots, v^n\}$

Natural candidates for $b: k \rightarrow V \otimes V^*$ and $d: V^* \otimes V \rightarrow k$ are

$$b(v) := \sum_i v_i \otimes v^i$$

and

$$d(v_i \otimes v^j) := \langle v^i, v^j \rangle = \delta_{ij}$$

Since they clearly satisfy $b^2 = 1$ and $d^2 = 1$.

However, to make sure that $b$ and $d$ are morphisms in $A$-Mod$_F$, we equip $\text{Hom}(V, k)$ with the $A$-action

$$\langle af, v^i \rangle := \langle f, S(a) v^i \rangle \quad \forall a \in A, f \in \text{Hom}(V, k) \quad (v \in V)$$

Now, $\text{Hom}(V, k)$ with this $A$-module structure is in $\text{Ob}(A$-Mod$_F$), denoted by $V^*$. Furthermore if $S$ is invertible, we can similarly define

$$b'(v) : k \rightarrow V \otimes V$$

and

$$d'(v_i \otimes v^j) := \langle v^j, v_i \rangle = \delta_{ij}$$

with the $A$-action (which makes $b'$ and $d'$ into $A$-module morphisms)

$$\langle af, v^i \rangle := \langle f, S^{-1}(a) v^i \rangle$$