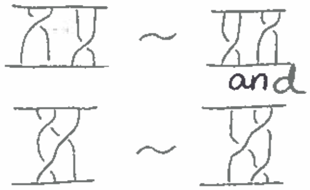
Braided Category ⇝ Invariant of braids

Last time, we introduced braids as special cases of tangles.
Now, let's write down explicitly the generators and relations for isotopy classes of braids (both as groups and as a tensor category).

The Braid Group on n strands B_n

As a group, B_n is generated by $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ (hence $B_n \ni \sigma_1^{-1}, \cdots, \sigma_{n-1}^{-1}$)
with the relations (when $n \geq 3$)

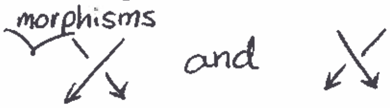


$\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$ and
$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$\Big\}$ for $1 \leq i, j \leq n-1$

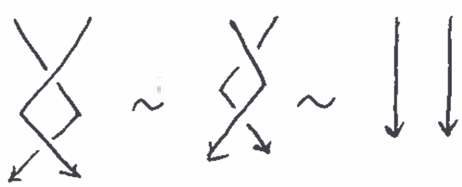
The Braid Category $\mathcal{B}$

As a strict tensor category (same setup as for the Tangle Category),
$\mathcal{B}$ is generated by morphisms

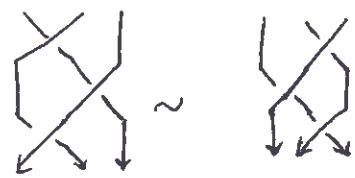


together with the relations

(RII)



and



(RIII)

Comment:

(1) We can deduce the generators and relations for $\mathcal{B}$ from the definition of B_n ($\forall n$):

we save on the # of generators ∵ we can tensor them with ↓ (identity) freely.

we need to add $\begin{cases} \text{} & \text{as a generator} \\ \text{(RII)} & \text{as a relation} \end{cases}$  ∵ unlike in a group where if $\sigma \in G$ then automatically $\sigma^{-1} \in G$.

(2) Compare generators & relations for $\mathcal{B}$ with those for $\mathcal{T}$:
exactly the ones without turns (∪ or ∩) survived.

Invariant of Braids

is a strict braided tensor functor from the Braid Category to another strict braided category.

can be relaxed

Braided Tensor Functor $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$

$\mathcal{C}$ and $\mathcal{D}$ are braided tensor categories

$(F, \varphi_0, \varphi_2)$ is a tensor functor from $\mathcal{C}$ to $\mathcal{D}$

s.t. for any pair (V, V') of objects in $\mathcal{C}$,

$$\begin{tikzcd} F(V)\otimes F(V') \arrow[r, "\varphi_2"] \arrow[d, "c_{F(V),F(V')}"'] & F(V\otimes V') \arrow[d, "F(c_{V,V'})"] \\ F(V')\otimes F(V) \arrow[r, "\varphi_2"] & F(V'\otimes V) \end{tikzcd}$$

φ_2 respects braiding c

Commutes.

Comment: If F is strict, then $c_{F(V),F(V')} = F(c_{V,V'})$. braiding c is preserved by F.

Braided Category $\mathcal{C} \xrightarrow{\text{picking vector}}$ Invariant of Braids

Given a braided category $\mathcal{C}$ and an object V of $\mathcal{C}$, we want to construct a strict braided tensor functor from $\mathcal{B}$ to $\mathcal{C}$.

Define $F\big(\underset{\in Ob(\mathcal{B})}{(+)}\big) := \underset{\in Ob(\mathcal{C})}{V}$.

We can extend F uniquely to a strict braided tensor functor $\mathcal{B} \to \mathcal{C}$.
(Exercise: verify.)

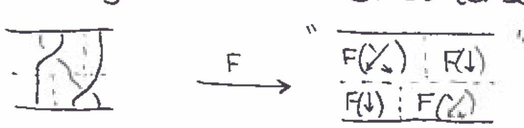
In particular, F() $= c_{V,V}$ and F() $= c_{V,V}^{-1}$.

$\in Hom(\mathcal{B})$ $\in Hom(\mathcal{C})$

To compute $F(B)$ for a given isotopy class of braids B:

① Write B as $\circ$ and $\otimes$ of the generating morphisms, say $B = \overset{m}{\underset{i=1}{\circ}}\big(\overset{n_i}{\underset{j=1}{\otimes}} B_{i,j}\big)$

② Then by def'n of a strict tensor functor, $F(B) = \overset{m}{\underset{i=1}{\circ}}\big(\overset{n_i}{\underset{j=1}{\otimes}} F(B_{i,j})\big)$

e.g.  $= (c_{V,V}\otimes id_V)\circ(id_V\otimes c_{V,V})$

Braided Hopf Algebra $\rightsquigarrow$ Braided Category with left duality
(if S is invertible) (also with right duality)

<u>Braided Hopf Algebra</u> $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ p175-176

is a braided bialgebra with invertible antipode.

Comment: At this point, I am not sure whether it's better to define braided Hopf algebras with an invertible antipode or just an antipode. So I will always say "braided Hopf algebra with an invertible antipode" if I want S to be invertible

<u>Strict Tensor Category with Left Duality</u> $(\mathcal{C}, \otimes, I, b, d)$ p342, 346
Right b' d'

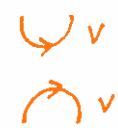
$(\mathcal{C}, \otimes, I)$ a strict tensor category

For each $V \in Ob(\mathcal{C})$, $\downarrow V$

Left Duality

$\exists$ a dual object $V^* \in Ob(\mathcal{C})$ $\uparrow V$ and

morphisms birth $b_V : I \to V \otimes V^*$
death $d_V : V^* \otimes V \to I$ $\in Hom(\mathcal{C})$



<u>s.t.</u> $(id_V \otimes d_V)(b_V \otimes id_V) = id_V$ and $(d_V \otimes id_{V^*})(id_{V^*} \otimes b_V) = id_{V^*}$

slightly ambiguous

Right Duality

$\exists$ a dual object ${}^*V \in Ob(\mathcal{C})$ $\uparrow V$ and

[Comment: ${}^*V$ is like a predual.]

morphisms $b'_V : I \to {}^*V \otimes V$
$d'_V : V \otimes {}^*V \to I$ $\in Hom(\mathcal{C})$

<u>s.t.</u> $(d'_V \otimes id_V)(id_V \otimes b'_V) = id_V$ and $(id_{*V} \otimes d'_V)(b'_V \otimes id_{*V}) = id_{*V}$

<u>Relation between Left & Right Duality</u>

If $\mathcal{C}$ is <u>autonomous</u>, i.e. it has both left & right duality, then ${}^*(V^*) = V = ({}^*V)^*$.

Strict Braided Category with Left/Right Duality

is a strict tensor category with left/right duality which is also braided.

Comment: As we saw in the definitions above,
the universal R-matrix and the antipode can be introduced separately for a bialgebra, so can
the braiding and duality in a tensor category.

We already saw how R gives rise to a braiding in A-Mod.
↖ universal R-matrix of a braided bialgebra A

We will now describe how S gives a duality structure to $A\text{-Mod}_f$.
antipode of Hopf alg. A ↗ finite-dim'l A-modules ↗

Construction: Antipode ⇝ Duality

p 347

Let A be a Hopf algebra with antipode S.

For $V \in A\text{-Mod}_f$ with a basis $\{v_1, \dots, v_n\}$,
consider its dual vector space $\mathrm{Hom}(V,k)$ with the dual basis $\{v^1, \dots, v^n\}$ → the underlying v.sp. for V^* ($\in Ob(A\text{-Mod}_f)$)

Natural candidates for $b_V: k \to V \otimes V^*$ and $d_V: V^* \otimes V \to k$ are

$$\boxed{b_V(1) := \sum_i v_i \otimes v^i} \quad \text{and} \quad \boxed{d_V(v^i \otimes v_j) := \langle v^i, v_j \rangle = \delta_{ij}}$$

↖ evaluation

Since they clearly satisfy (zigzag) $= \downarrow$ and (zigzag) $= \uparrow$.

However, to make sure that b_V and d_V are morphisms in $A\text{-Mod}_f$,
we equip $\mathrm{Hom}(V,k)$ with the A-action

$$\boxed{\langle af, v \rangle := \langle f, S(a) v \rangle} \qquad \forall a \in A,\ f \in \mathrm{Hom}(V,k)\ (v \in V)$$

Now, $\mathrm{Hom}(V,k)$ with this A-module structure is in $Ob(A\text{-Mod}_f)$, denoted by V^*.

Furthermore if S is invertible, we can similarly define
$b'_V: k \to {}^*V \otimes V$ and $d'_V: V \otimes {}^*V \to k$ by

$$\boxed{b'_V(1) := \sum_i v^i \otimes v_i} \quad \text{and} \quad \boxed{d'_V(v_i \otimes v^j) = \langle v^j, v_i \rangle = \delta_{ij}}$$

(which clearly satisfy duality axioms)

where ${}^*V \in Ob(A\text{-Mod}_f)$ is the vector space $\mathrm{Hom}(V,k)$
with the A-action (which makes b'_V and d'_V into A-module morphisms)

$$\boxed{\langle af, v \rangle := \langle f, S^{-1}(a) v \rangle}.$$