Module $\leftrightarrow$ Comodule Duality (over finite-dim' algebra/coalgebra)

\[ V^* \xrightarrow{\Delta_{V^*}} V^* \otimes H^* \xrightarrow{\mu} V^* \]

\[ (V \otimes H)^* \xrightarrow{\Delta_{(V \otimes H)^*}} (V^* \otimes H^*) \]

\[ \bar{\lambda}(a \otimes b) = a(a) \cdot \beta(b) \]

\( \bar{\lambda} \) is an isomorphism of vector spaces

\[ \Leftrightarrow \text{H or V is finite-dim'}. \]

\[ \begin{array}{c}
\text{H left-comodule } (V, \Delta_V) \xrightarrow{\text{H}} \text{H* right-module } (V^*, \nu^* \mu := \Delta_{V^*}^* \bar{\lambda}) \\
\text{H right-module } (V, \nu \mu) \xrightarrow{\text{H*}} \text{H* left-comodule } (V^*, \Delta_{V^*} = \bar{\lambda}^{-1} \nu^* \mu^*)
\end{array} \]
Categorical Language

Motivations:
1. Categorical description of modules over algebras provides a link between algebra and topological invariants.
2. Categorical description leads to generalization of the algebraic objects.

Algebras
- braided bialgebra
  - \( s \)
- braided Hopf algebra
  - \( \theta \)
- ribbon algebra
  - can relax to quasi

Categories
- braided category
  - dual
  - braided category with twist
  - twist
  - ribbon category
  - all are tensor categories

Topological Invariants
- "Invariant" of braids
  - \( \eta \)
  - "Invariant" of framed braids
  - invariant of tangles/links
    - framing
  - invariant of ribbons/framed links

4.6
Category: \( C = (\text{Ob}(C), \text{Hom}(C); \, s, b, o, \text{id}) \)

- \( \text{Ob}(C) \): the class of objects of \( C \)
- \( \text{Hom}(C) \): the class of morphisms (between objects) of \( C \)

4 maps:
- Source: \( s: \text{Hom}(C) \rightarrow \text{Ob}(C) \)
- Target: \( b: \text{Hom}(C) \rightarrow \text{Ob}(C) \)
- Composition: \( o: \text{Hom}(C) \times \text{Ob}(C) \rightarrow \text{Hom}(C) \)
- Identity: \( \text{id}: \text{Ob}(C) \rightarrow \text{Hom}(C) \)

\( \text{Hom}(C) \times \text{Ob}(C) \rightarrow \text{Hom}(C) \)

- \( s \) is a surjective homomorphism
- \( b \) is a surjective homomorphism

Composition is associative

\( s(f \circ g) = s(f) \cdot s(g) \)

\( \text{id}(f) = f \)

For \( f \in \text{Hom}(C) \), let \( V := s(f) \), \( W := b(f) \).

Then we also write \( f: V \rightarrow W \).

Functor (between categories): \( F: C \rightarrow C' \)

- \( F: \text{Ob}(C) \rightarrow \text{Ob}(C') \)
- \( \text{Hom}(C) \rightarrow \text{Hom}(C') \)

\( \text{Isomorphism} \) is a family of morphisms in \( C' \) indexed by objects of \( C \)

\( \text{Iso}(C) \)

Natural Transformation (between functors): \( \eta: F \rightarrow G \) (where \( F, G: C \rightarrow C' \))

\( \text{Isomorphism} \) of objects \( \eta: \text{Ob}(C) \rightarrow \text{Hom}(C') \)

\( \text{Isomorphism} \) of objects

\( \eta: \text{Ob}(C) \rightarrow \text{Hom}(C') \)

\( \text{Isomorphism} \) of objects
Tensor Category \((C, \otimes, I, a, l, r)\)

\[ C = (\text{Obj}(C), \text{Hom}(C); s, b, \circ, \text{id}) \] a category

\(\otimes\) a functor from \(C \times C \to C\)

More explicitly written out:

\((V, W) \in \text{Obj}(C) \times \text{Obj}(C) \mapsto V \otimes W \in \text{Obj}(C)\)

\((f, g) \in \text{Hom}(C) \times \text{Hom}(C) \mapsto f \circ g \in \text{Hom}(C)\)

\[ S(f \circ g) = S(f) \otimes S(g) \]

\[ b(f \circ g) = b(f) \otimes b(g) \]

\[ (f' \circ g') \circ (f \circ g) = (f' \circ f) \otimes (g' \circ g) \]

\[ \text{id}_{V \otimes W} = \text{id}_V \otimes \text{id}_W \]

**Unit** \(I \in \text{Obj}(C)\) is a special object of \(C\) (see \(l\) and \(r\))

**Associative Constraint**

\[ a_{V, W, X} : (V \otimes W) \otimes X \to V \otimes (W \otimes X) \]

**Left Constraint**

\[ l_V : I \otimes V \cong V \]

\[ l_V \circ \text{id}_{I \otimes f} = f \]

**Right Constraint**

\[ r_V : V \otimes I \cong V \]

\[ r_V \circ \text{id}_V \otimes f = f \]

**Pentagon Axiom**

**Triangle Axiom**